Acta Cryst. (1994). A**50**, 403–405

Proportional Colored Symmetry*

BY SLAVIK V. JABLAN

The Mathematical Institute, 11001 *Belgrade*, *Knez Mihailova* 35, *pp* 367, *Yugoslavia*

(*Received* 26 *January* 1993; *accepted* 27 *September* 1993)

Abstract

Proportional colored symmetry is defined and the corresponding proportional colored-symmetry structures are derived from Belov (p)-symmetry plane groups.

The concept of P-symmetry (permutation symmetry), introduced by Zamorzaev (Zamorzaev, Galyarskij & Palistrant, 1978; Zamorzaev, Karpova, Lungu & Palistrant, 1986), is defined as follows. If P is a permutation group under the set $C=\{1,2,\ldots,p\}$ and G is a discrete symmetry group, every transformation $S=cs=sc$, $c\in P$ and $s\in G$, is a P-symmetry transformation. Every group G^P, derived from G by substitution of symmetries by P-symmetries, is a P-symmetry group. If the substitutions included in G^P exhaust the group P, G^P is a complete P-symmetry group. Every complete P-symmetry group G^P can be derived from its generating group G by searching in G and P for the normal subgroups H and Q, for which the isomorphism $G/H \cong P/Q$ holds, by paired multiplication of the cosets corresponding in this isomorphism and by the unification of the products obtained. The groups of complete P-symmetry fall into senior ($G=H$ and $G^P = G\times P$), junior ($G/H\cong P$ and $G^P\cong G$) and middle groups for $Q=P$, $Q=I$ and $I\subset Q\subset P$, respectively. In this paper, only the junior P-symmetry groups of complete P-symmetry will be considered.

Let P be a permutation group under the set $C=\{1,2,\ldots,p\}$. C' is a coloring of the set C by k colors if to every i ($i\in C$) is linked an index j ($j\in\{1,2,\ldots,k\}$) denoting certain properties (*e.g.* color). A coloring is termed complete if all k colors ($k\le p$) are used. From every coloring C' the corresponding group P' is induced. Two colorings, C' and C'', of the same set C with k colors are equivalent with regard to the group P if there is a color-preserving permutation, $f\in P$ [$f(i_j)=f(i)_j$] such that $f(C')=C''$. The coloring C' is termed symmetrical (or reducible) if there is a color-preserving non-identical permutation $f\in P$ such that $f(C')=C'$.

In order to find all the colorings of the set C, which are non-equivalent with regard to a specific permutation group P, it is appropriate to use the geometrical scheme corresponding to the P-symmetry considered. For example, Belov P-symmetry with the permutation group $P\cong C_p$, generated by the cyclic permutation $c=(1\ 2\ \ldots\ p)$, can be modeled by a regular oriented p-gon. Pawley (P')-symmetry with the regular dihedral permutation group $P\cong D_{p_{(2p)}}$, generated by the permutations $c=(1\ 2\ \ldots\ p)(2p\ 2p-1\ \ldots\ p+1)$ and $c_1=(1\ p+1)(2\ p+2)\ldots(p\ 2p)$, can be modeled by a truncated regular p-gon *etc.*

According to the geometrical classification of P-symmetries (Zamorzaev & Palistrant, 1981; Zamorzaev, Karpova, Lungu & Palistrant, 1986), the group P is isomorphic to the discrete crystallographic point group P_0. In order to avoid the crystallographic restriction, we may accept that the group P is isomorphic to the point group P_0 belonging to one of the seven infinite classes of point groups: n, $(\widetilde{2n})$, $n{:}m$, nm, $n{:}2$, $(\widetilde{2n})m$ and $mn{:}m$. Here, every P-symmetry possesses the corresponding geometrical scheme.

The next concept to consider is the W-symmetry introduced by Koptsik & Kotsev (Koptsik & Kotsev, 1974; Koptsik, 1988). Since the idea of W-symmetry is closely connected to the problem of different colorings of a symmetrical figure (Zamorzaev, Karpova, Lungu & Palistrant, 1986), for simplification, this approach will be accepted.

If M is a point in a general position with regard to the discrete finite symmetry group P_0, to each point of the orbit $P_0(M)=\{g(M)|g\in P_0\}$ can correspond one from k different properties (*e.g.* colors). Two colorings of the orbit corresponding to the symmetry group P_0 are equal if the symmetry of the orbit transforms the first colored orbit into the second without changing the colors of that orbit points. The coloring of the orbit is irreducible if and only if the colored orbit is asymmetrical. With the corresponding geometrical schemes for the P-symmetries, non-equivalent orbit colorings are simply visible.

The concept of proportional colored symmetry originated from Grünbaum, Grünbaum & Shephard (1986). Let G be a plane symmetry group, F a figure

* This research was supported by Science Fund of Serbia, grant no. 0401A, through Matematički Institut.

satisfying it, G^P the P-symmetry group derived from G with the permutation group P under the set C (isomorphic to a discrete point group P_0), F^P a figure with the P-symmetry group G^P, C' a coloring of the set C by k colors, P' a group induced by the coloring C' and $F^{P'}$ the induced coloring of the figure F^P. The colored-symmetry structure F', obtained from $F^{P'}$ by the transformation $i_j \to j$, is a proportional colored symmetry structure. If two proportional symmetry structures are equivalent, there is an affine transformation converting one structure into the other, without changing the colors. In order to find all the non-equivalent proportional colored-symmetry structures derived from a plane symmetry group G, we can use the following algorithm: (*a*) find all the P-symmetry groups derived from the symmetry group G, with the group P fixed; (*b*) find all the non-equivalent colorings of the set C with regard to the group P; (*c*) from the induced figures $F^{P'}$ resulting from the groups (*a*) and colorings (*b*), derive, using the transformation $i_j \to j$, all the non-equivalent proportional colored symmetry structures F'. The term 'proportional colored symmetry structures' denotes that the proportion of colors used for the coloring C' is extended to the figure F'. Also, among all the non-equivalent colorings, we can distinguish the colorings with the same proportion of colors.

In this paper, we will restrict our attention to the proportional colored symmetry structures derived from Belov's (P)-symmetry plane groups. For $p = 2$, there are 46 antisymmetry groups, giving 46 complete proportional symmetry structures with a 1:1 proportion of colors. Except for $p = 3$, where six additional (3)-symmetry groups exist: $\{a,b\}(3^{(3)})$, $\{a,b\}(3^{(-3)})$, $\{a^{(3)},b^{(3)}\}(3^{(3)})$, $\{a^{(3)},b^{(3)}\}(m3)$, $\{a,b\}(6^{(3)})$, $\{a,b\}(6^{(-3)})$; $p = 4$, where six additional (4)-symmetry groups exist: $\{a^{(2)},b^{(2)}\}(2b/2m^{(4)}_{a/4})$, $\{a,b\}(4^{(4)})$, $\{a,b\}(4^{(-4)})$, $\{a^{(2)},b^{(2)}\}(4^{(4)})$, $\{a^{(2)},b^{(2)}\}(4^{(4)}b/2m^{(4)}_{a/4})$, $\{a^{(2)},b^{(2)}\}(4^{(4)}b/2m^{(-4)}_{a/4})$; and $p = 6$, where three additional (6)-symmetry groups exist: $\{a^{(3)},b^{(3)}\}(m^{(2)}3)$, $\{a,b\}(6^{(6)})$; $\{a,b\}(6^{(-6)})$; for every $p = 2n + 1$ ($n \in N$), $2\varphi(p)$ (p)-symmetry groups exist: $\{a^{(p/q)},b\}$, $\{a,b^{(p/q)}\}(m)$, $\{a,b^{(p/2q)}\}(b/2m^{(p/q)})$, $\{a,(a+b)/2^{(p/q)}\}(m)$, where φ is Euler's function; for every $p = 4n$ ($n \in N$), $4\varphi(p)$ (p)-symmetry groups exist: $\{a^{(p/q)},b\}$, $\{a,b^{(p/q)}\}(m)$, $\{a,b^{(p/q)}\}(m^{(2)})$, $\{a^{(2)},b^{(p/q)}\}(m)$, $\{a,b^{(p/2q)}\}$-$(b/2m^{(p/q)})$, $\{a^{(2)},b^{(p/2q)}\}(b/2m^{(p/q)})$, $\{a,(a+b)/2^{(p/q)}\}(m)$, $\{a,(a+b)/2^{(p/q)}\}(m^{(2)})$; and for $p = 4n + 2$ ($n \in N$), $11\varphi(p)/2$ (p)-symmetry groups exist: $\{a^{(p/q)},b\}$, $\{a,b^{(p/q)}\}(m)$, $\{a,b^{(p/2q)}\}(m^{(2)})$, $\{a,b^{(p/q)}\}(m^{(2)})$, $\{a^{(2)},b^{(p/2q)}\}(m)$, $\{a^{(2)},b^{(p/q)}\}(m)$, $\{a,b^{(p/2q)}\}(b/2m^{(p/q)})$, $\{a^{(2)},b^{(p/2q)}\}(b/2m^{(p/q)})$, $\{a,(a+b)/2^{(p/q)}\}(m)$, $\{a,(a+b)/2^{(p/2q)}\}(m^{(2)})$, $\{a,(a+b)/2^{(p/q)}\}(m^{(2)})$. Knowing this, in order to derive all the non-equivalent proportional colored symmetry structures from the Belov (p)-symmetry plane groups, we only need to find the number of different irreducible colorings of a regular oriented p-gon by k colors.

Theorem. The number of different colorings of a regular oriented p-gon by k colors, p_k', is given by the formula

$$p_k' = (1/p)\sum_{q|p} \varphi(p/q)k^q,$$

where φ is Euler's function, defined as $\varphi(m) = m \times \Pi^n_{i=1} (1 - 1/p_i)$, and m is given by the product of the distinct prime numbers $p_1, p_2, \ldots, p_n$ as $m = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$. The sum is then taken for all q divisors of p (*i.e.* for all q, $q|p$).

Euler's function, $\varphi(m)$, can also be defined as the number of natural numbers not exceeding m, which are mutually prime with m. Amongst them, there are $|p_k|' = p_k' - \sum_{i=1}^{k-1} \binom{k}{i} |p_i|'$ complete colorings, where $\binom{k}{i}$ is the binomial coefficient $\binom{k}{i} = k!/(k-i)!i!$.

Among the $|p_k|'$ complete colorings, there are

$$(p_k)' = |p_k|' - \sum_{\substack{q|p \\ 1<q<p}} (q_k)'$$

complete irreducible colorings.

As an example of proportional colored-symmetry structures derived from Belov (p)-symmetry plane groups, for the crystallographic values of $p = 3$, 4, 6 and $k = 2$, we have 20, 42 and 126, respectively, complete irreducible proportional colored-symmetry structures. Every proportional colored-symmetry structure can be denoted by a symbol (G^P,P'). For

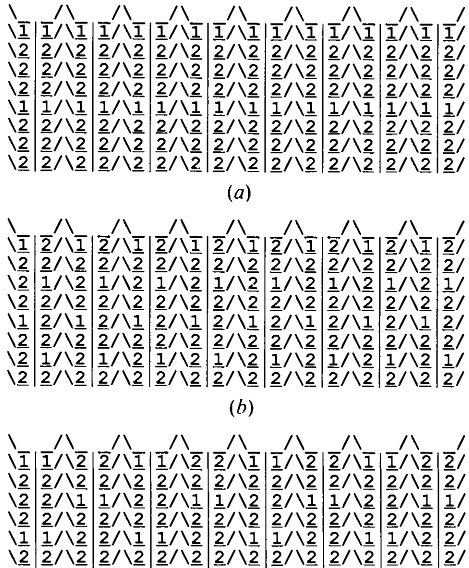
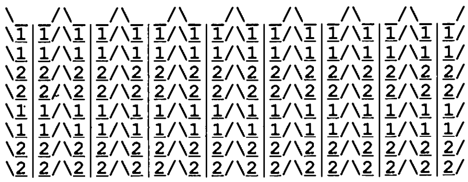
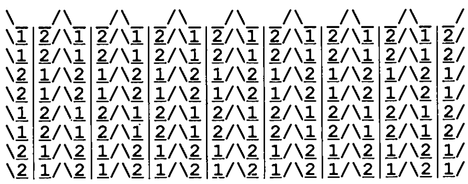


Fig. 1. Irreducible proportional symmetry structures with a 1:3 proportion of colors: (*a*) ($\{a,b^{(4)}\}(m)$, $\{(1_1\ 2_2\ 3_2\ 4_2)\}$), (*b*) ($\{a,b^{(4)}\}(m^{(2)})$, $\{(1_2\ 2_2\ 3_2\ 4_2)\}$) and (*c*) ($\{a^{(2)},b^{(4)}\}(m)$, $\{(1_1\ 2_2\ 3_2\ 4_2)\}$).

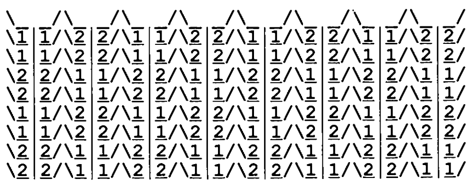
example, from the symmetry group $\{a,b\}(m)$ and $P = C_4 \cong \{(1\ 2\ 3\ 4)\}$ are derived nine complete irreducible proportional symmetry structures: $(\{a,b^{(4)}\}(m), \{(1_1\ 2_2\ 3_2\ 4_2)\})$, $(\{a,b^{(4)}\}(m^{(2)}), \{(1_1\ 2_2\ 3_2\ 4_2)\})$, $(\{a^{(2)},b^{(4)}\}(m), \{(1_1\ 2_2\ 3_2\ 4_2)\}]$ with a 1:3 proportion of colors (Fig. 1); $(\{a,b^{(4)}\}(m), \{(1_1\ 2_1\ 3_2\ 4_2)\})$, $(\{a,b^{(4)}\}(m^{(2)}), \{(1_1\ 2_1\ 3_2\ 4_2)\})$, $(\{a^{(2)},b^{(4)}\}(m), \{(1_1\ 2_1\ 3_2\ 4_2)\})$ with a 2:2 proportion of colors (Fig. 2); and $(\{a,b^{(4)}\}(m), \{(1_1\ 2_1\ 3_1\ 4_2)\})$, $(\{a,b^{(4)}\}(m^{(2)}), \{(1_1\ 2_1\ 3_1\ 4_2)\})$, $(\{a^{(2)},b^{(4)}\}(m), \{(1_1\ 2_1\ 3_1\ 4_2)\})$ with a 3:1 proportion of colors. We notice that the proportional colored-symmetry structures derived from the same P-symmetry groups, with the group $P' = \{(1_1\ 2_2\ 3_1\ 4_2)\}$ and a 2:2 proportion of colors, are not included in the list because the coloring of the regular oriented 4-gon resulting in the group $P' = \{(1_1\ 2_2\ 3_1\ 4_2)\}$ is reducible and the structures obtained belong to the class of antisymmetry colorings with a 1:1 proportion of colors.



(a)



(b)



(c)

Fig. 2. Irreducible proportional symmetry structures with a 2:2 proportion of colors: (a) $(\{a,b^{(4)}\}(m), \{(1_1\ 2_1\ 3_2\ 4_2)\})$, (b) $(\{a,b^{(4)}\}(m^{(2)}), \{(1_1\ 2_1\ 3_2\ 4_2)\})$ and (c) $(\{a^{(2)},b^{(4)}\}(m), \{(1_1\ 2_1\ 3_2\ 4_2)\})$.

Since in the theory of P-symmetry general results for the P-symmetry plane groups are not complete (Zamorzaev, Galyarskij & Palistrant, 1978; Zamorzaev, Karpova, Lungu & Palistrant, 1986; Wieting, 1982), as well as in the theory of different colorings of symmetrical figures, further progress in the derivation of proportional colored-symmetry structures, solely represented by the specific combination of non-positional symmetry (P-symmetry) and positional symmetry (W-symmetry), will be conditioned by the future development of both fields. The idea of proportional coloring of crystallographic structures may be useful in the classification schemes of P- and W-symmetry structures.

References

GRÜNBAUM, B., GRÜNBAUM, Z. & SHEPHARD, G. C. (1986). *Comput. Math. Appl.* **12B**, 3–4, 641–653.

KOPTSIK, V. A. (1988). *Comput. Math. Appl.* **16**, 5–8, 407–424.

KOPTSIK, V. A. & KOTSEV, I. N. (1974). *K Teorii i Klasifikatsii Grupp Tsvetnoj Simmetrii.* II. *W-Simmetriya.* Soobscheniya OIYaI P4-8068, Dubna, Russia.

WIETING, T. W. (1982). *The Mathematical Theory of Chromatic Plane Ornaments.* New York/Basel: Marcel Dekker.

ZAMORZAEV, A. M., GALYARSKIJ, E. I. & PALISTRANT, A. F. (1978). *Tsvetnaya Simmetriya, Eyo Obobscheniya i Prilozeniya.* Kishinev: Shtiintsa.

ZAMORZAEV, A. M., KARPOVA, YU. S., LUNGU, A. P. & PALISTRANT, A. F. (1986). *P-Simmetriya i Eyo Dalnejshee Razvitie.* Kishinev: Shtiintsa.

ZAMORZAEV, A. M. & PALISTRANT, A. F. (1981). *Dokl. Akad. Nauk SSSR*, **256**, 856–859.